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A CORRESPONDENCE OF CANONICAL BASES IN THE q-DEFORMED HIGHER LEVEL FOCK SPACES

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ABSTRACT. The q-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The q-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the q-deformed Fock space. In this paper, we show that parts of q-decomposition matrices of level ℓ coincides with that of level $\ell-1$ under certain conditions of multi charge.

1. INTRODUCTION

The q-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s=(s_1,\ldots,s_\ell)\in\mathbb{Z}^\ell$, the q-deformed Fock space $F_q[s]$ of level ℓ is the $\mathbb{Q}(q)$-vector space whose basis are indexed by ℓ-tuples of Young diagrams. i.e. $\{|\lambda;s\rangle\mid\lambda\in\Pi^\ell\}$, where Π is the set of Young diagrams.

The canonical bases $\{G^+(\lambda;s)\mid\lambda\in\Pi^\ell\}$ and $\{G^-(\lambda;s)\mid\lambda\in\Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $^-$ [Ugl00]. Define matrices $\Delta^+(q)=(\Delta^+_{\lambda,\mu}(q))_{\lambda,\mu}$ and $\Delta^-(q)=(\Delta^-_{\lambda,\mu}(q))_{\lambda,\mu}$ by

$$G^+(\lambda;s)=\sum_\mu\Delta^+_{\lambda,\mu}(q)\,|\mu;s\rangle\quad,\qquad G^-(\lambda;s)=\sum_\mu\Delta^-_{\lambda,\mu}(q)\,|\mu;s\rangle.$$

We call $\Delta^+_{\lambda,\mu}(q)$ and $\Delta^-_{\lambda,\mu}(q)$ *q-decomposition numbers*. These q-decomposition matrices plays an important role in representation theory. However it is difficult to compute q-decomposition matrices.

In the case of $\ell=1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(q)$ coincides with the decomposition matrix of v-Schur algebra. For $\ell\geq 2$, Yvonne [Yvo07] conjectured that the matrix $\Delta^+(q)$ coincides with the q-analogue of the decomposition matrix of cyclotomic Schur algebras at a primitive n-th root of unity under a suitable condition of multi charge. Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in the category O of rational Cherednik algebras are equal to the corresponding coefficients $\Delta^+_{\lambda,\mu}(q)$.

We say that the j-th component s_j of the multi charge is *sufficiently large* for $|\lambda;s\rangle$ if $s_j-s_i\geq\lambda^{(i)}_1$ for any $i=1,2,\cdots,\ell$, and that s_j is *sufficiently small* for $|\lambda;s\rangle$ if $s_i-s_j\geq|\lambda|=|\lambda^{(1)}|+\cdots+|\lambda^{(\ell)}|$ for any $i=1,2,\cdots,\ell$ (see Definition 3.1). If s_j is sufficiently large for $|\lambda;s\rangle$ and $|\lambda;s\rangle>|\mu;s\rangle$, then the j-th components of λ and μ are both the empty Young diagram $\emptyset$ (Lemma 3.2). On the other hand, if s_j is sufficiently small for $|\lambda;s\rangle$ and $|\lambda;s\rangle\geq|\mu;s\rangle$, then $\mu^{(j)}=\emptyset$ implies $\lambda^{(j)}=\emptyset$. (Lemma 3.3).

Our main results are as follows.

Theorem A. (Theorem 3.4) [Iij]
Let $\varepsilon\in\{+,-\}$. If s_j is sufficiently large for $|\lambda;s\rangle$, then

$$\Delta^\varepsilon_{\lambda,\mu;s}(q)=\Delta^\varepsilon_{\check{\lambda},\check{\mu};\check{s}}(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j-th component of λ (resp. μ, s), $\Delta^{\varepsilon}_{\lambda,\mu;s}(q)$ is the q-decomposition number of level ℓ and $\Delta^{\varepsilon}_{\check{\lambda},\check{\mu};\check{s}}(q)$ is the q-decomposition number of level $\ell - 1$.

Theorem B. (Theorem 3.5) [Iij]
Let $\varepsilon \in \{+,-\}$. If s_j is sufficiently small for $|\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then

$$\Delta^{\varepsilon}_{\lambda,\mu;s}(q) = \Delta^{\varepsilon}_{\check{\lambda},\check{\mu};\check{s}}(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j-th component of λ (resp. μ, s).

This paper is organized as follows. In Section 2, we review the q-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results.

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Notations. For a positive integer N, a *partition* of N is a non-increasing sequence of non-negative integers summing to N. We write $|\lambda| = N$ if λ is a partition of N. The *length* $l(\lambda)$ of λ is the number of non-zero components of λ. And we use the same notation λ to represent the Young diagram corresponding to λ. For an ℓ-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(\ell)}|$.

2. The q-deformed Fock spaces of higher levels

2.1. q-wedge products and straightening rules. Let n, ℓ, s be integers such that $n \geq 2$ and $\ell \geq 1$. We define $P(s)$ and $P^{++}(s)$ as follows;

$$P(s) = \{\boldsymbol{k} = (k_1, k_2, \cdots) \in \mathbb{Z}^{\infty} \mid k_r = s - r + 1 \text{ for any sufficiently large } r\}, \tag{1}$$

$$P^{++}(s) = \{\boldsymbol{k} = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots\}. \tag{2}$$

Let Λ^s be the $\mathbb{Q}(q)$ vector space spanned by the q-wedge products

$$u_{\boldsymbol{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \quad , \quad (\boldsymbol{k} \in P(s)) \tag{3}$$

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on n and ℓ. [Ugl00, Proposition 3.16].

Example 2.1. (*i*) *For every* $k_1 \in \mathbb{Z}$, $u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. *Therefore* $u_{k_1} \wedge u_{k_1} = 0$.
(*ii*) *Let* $n = 2$, $\ell = 2$, $k_1 = -2$, *and* $k_2 = 4$. *Then*

$$u_{-2} \wedge u_4 = q\, u_4 \wedge u_{-2} + (q^2 - 1)\, u_2 \wedge u_0.$$

(*iii*) *Let* $n = 2$, $\ell = 2$, $k_1 = -1$, $k_2 = -2$ *and* $k_3 = 4$. *Then*

$$\begin{aligned} u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge (u_{-2} \wedge u_4) &= u_{-1} \wedge \left(q\, u_4 \wedge u_{-2} + (q^2 - 1)\, u_2 \wedge u_0\right) \\ &= q\, u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1)\, u_{-1} \wedge u_2 \wedge u_0 \end{aligned}$$

By applying the straightening rules, every q-wedge product $u_{\boldsymbol{k}}$ is expressed as a linear combination of so-called *ordered q-wedge products,* namely q-wedge products $u_{\boldsymbol{k}}$ with $\boldsymbol{k} \in P^{++}(s)$. The ordered q-wedge products $\{u_{\boldsymbol{k}} \mid \boldsymbol{k} \in P^{++}(s)\}$ form a basis of Λ^s called *the standard basis*.

2.2. **Abacus.** It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with N runners labeled $1, 2, \cdots N$ from left to right. The positions on the i-th runner are labeled by the integers having residue i modulo N.

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
-N+1 & -N+2 & \cdots & -1 & 0 \\
1 & 2 & \cdots & N-1 & N \\
N+1 & N+2 & \cdots & 2N-1 & 2N \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Each $\boldsymbol{k} \in P^{++}(s)$ (or the corresponding q-wedge product $u_{\boldsymbol{k}}$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions $k_1, k_2, \cdots$. We call this configuration *the abacus presentation* of $u_{\boldsymbol{k}}$.

Example 2.2. *If $n = 2$, $\ell = 3$, $s = 0$, and $\boldsymbol{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots)$, then the abacus presentation of $u_{\boldsymbol{k}}$ is*

$$\begin{array}{cc|cc|cc|l}
d=1 & & d=2 & & d=3 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
-17 & -16 & -15 & -14 & -13 & -12 & \cdots m=3 \\
-11 & -10 & -9 & -8 & -7 & -6 & \cdots m=2 \\
-5 & -4 & -3 & -2 & -1 & 0 & \cdots m=1 \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots m=0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
c=1 & c=2 & c=1 & c=2 & c=1 & c=2 &
\end{array}$$

We use another labeling of runners and positions. Given an integer k, let c, d and m be the unique integers satisfying

(4) $$k = c + n(d-1) - n\ell m \quad , \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.$$

Then, in the abacus presentation, the position k is on the $c + n(d-1)$-th runner (see the previous example). Relabeling the position k by $c - nm$, we have ℓ abaci with n runners.

Example 2.3. *In the previous example, relabeling the position k by c − nm, we have*

$$\begin{array}{cc|cc|cc|l}
d=1 & & d=2 & & d=3 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
-5 & -4 & -5 & -4 & -5 & -4 & \cdots m=3 \\
-3 & -2 & -3 & -2 & -3 & -2 & \cdots m=2 \\
-1 & 0 & -1 & 0 & -1 & 0 & \cdots m=1 \\
1 & 2 & 1 & 2 & 1 & 2 & \cdots m=0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
c=1 & c=2 & c=1 & c=2 & c=1 & c=2 &
\end{array}$$

We assign to each of ℓ abacus presentations with n runners a q-wedge product of level 1. In fact, straightening rules in each "sector" are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see Example 2.5 below)

We introduce some notation.

Definition 2.4. *For an integer k, let c, d and m be the unique integers satisfying (4), and write*

$$u_k = u^{(d)}_{c-nm}. \tag{5}$$

Also we write $u^{(d_1)}_{c_1-nm_1} > u^{(d_2)}_{c_2-nm_2}$ if $k_1 > k_2$, where $k_i = c_i + n(d_i - 1) - n\ell m_i$, $(i = 1, 2)$.

We regard $u^{(d)}_{c-nm}$ as u_{c-nm} in the case of $\ell = 1$.

Example 2.5. *If $n = 2$, $\ell = 3$, then we have*

$$u_{-10} \wedge u_1 = -q^{-1} u_1 \wedge u_{-10} + (q^{-2} - 1) u_{-4} \wedge u_{-5},$$

that is,

$$u^{(1)}_{-2} \wedge u^{(1)}_1 = -q^{-1} u^{(1)}_1 \wedge u^{(1)}_{-2} + (q^{-2} - 1) u^{(1)}_0 \wedge u^{(1)}_{-1}.$$

On the other hand, in the case of $n = 2$, $\ell = 1$,

$$u_{-2} \wedge u_1 = -q^{-1} u_1 \wedge u_{-2} + (q^{-2} - 1) u_0 \wedge u_{-1}.$$

2.3. **ℓ-tuples of Young diagrams.** Another indexation of the ordered q-wedge products is given by the set of pairs $(\boldsymbol{\lambda}, \boldsymbol{s})$ of ℓ-tuples of Young diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(\ell)})$ and integer sequences $\boldsymbol{s} = (s_1, \cdots, s_\ell)$ summing up to s. Let $\boldsymbol{k} = (k_1, k_2, \cdots) \in P^{++}(s)$, and write

$$k_r = c_r + n(d_r - 1) - n\ell m_r \quad , \quad 1 \le c_r \le n \quad , \quad 1 \le d_r \le \ell \quad , \quad m_r \in \mathbb{Z} \quad .$$

For $d \in \{1, 2, \cdots, \ell\}$, let $k^{(d)}_1, k^{(d)}_2, \cdots$ be integers such that

$$\beta^{(d)} = \{c_r - nm_r \mid d_r = d\} = \{k^{(d)}_1, k^{(d)}_2, \cdots\} \quad \text{and} \quad k^{(d)}_1 > k^{(d)}_2 > \cdots$$

Then we associate to the sequence $(k^{(d)}_1, k^{(d)}_2, \cdots)$ an integer s_d and a partition $\lambda^{(d)}$ by

$$k^{(d)}_r = s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda^{(d)}_r = k^{(d)}_r - s_d + r - 1 \quad \text{for } r \ge 1.$$

In this correspondence, we also write

$$u_{\boldsymbol{k}} = |\boldsymbol{\lambda}; \boldsymbol{s}\rangle \quad (\boldsymbol{k} \in P^{++}(s)). \tag{6}$$

Example 2.6. *If $n = 2$, $\ell = 3$, $s = 0$, and $\boldsymbol{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots)$, then*

$$k_1 = 6 = 2 + 2(3 - 1) - 6 \cdot 0 \quad , \quad k_2 = 3 = 1 + 2(2 - 1) - 6 \cdot 0 \quad ,$$
$$k_3 = 2 = 2 + 2(1 - 1) - 6 \cdot 0 \quad , \cdots \text{ and so on.}$$

Hence,

$$\beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\} \quad , \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\} \quad , \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\} \quad .$$

Thus, $\boldsymbol{s} = (2, 0, -2)$ and $\boldsymbol{\lambda} = (\emptyset, (1, 1), (4))$.

Note that we can read off $\boldsymbol{s} = (2, 0, -2)$ and $\boldsymbol{\lambda} = (\emptyset, (1, 1), (4))$ from the abacus presentation. (*see Example 2.3*)

2.4. **The q-deformed Fock spaces of higher levels.**

Definition 2.7. *For $s \in \mathbb{Z}^\ell$, we define the q-deformed Fock space $F_q[s]$ of level ℓ to be the subspace of Λ^s spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^\ell$):*

$$F_q[s] = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}(q)\, |\lambda; s\rangle. \tag{7}$$

We call s a multi charge.

2.5. **The bar involution.**

Definition 2.8. *The involution $\overline{}$ of Λ^s is the $\mathbb{Q}$-vector space automorphism such that $\overline{q} = q^{-1}$ and*

$$\overline{u_k} = \overline{u_{k_1} \wedge \cdots \wedge u_{k_r}} \wedge u_{k_{r+1}} \wedge \cdots = (-q)^{\kappa(d_1,\cdots,d_r)} q^{-\kappa(c_1,\cdots,c_r)} (u_{k_r} \wedge \cdots \wedge u_{k_1}) \wedge u_{k_{r+1}} \wedge \cdots, \tag{8}$$

where c_i, d_i are defined by k_i as in (4), r is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \cdots, a_r)$ is defined by

$$\kappa(a_1, \cdots, a_r) = \#\{(i,j) \mid i < j,\ a_i = a_j\}.$$

Remarks (i) The involution is well defined. i.e. it doesn't depend on r [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra $\hat{H}_r$. (see [Ugl00] for more detail.)

(iii) The involution preserves the q-deformed Fock space $F_q[s]$ of higher level.

2.6. **The dominance order.** We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\tilde{\lambda}$ and $\tilde{\mu}$ as

$$\tilde{\lambda} = \{\lambda_a^{(d)} + s_d \mid 1 \leq d \leq \ell,\ 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\},$$
$$\tilde{\mu} = \{\mu_a^{(d)} + s_d \mid 1 \leq d \leq \ell,\ 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}.$$

We denote by $(\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots)$ (resp. $(\tilde{\mu}_1, \tilde{\mu}_2, \cdots)$) the sequence obtained by rearranging the elements in the multi-set $\tilde{\lambda}$ (resp. $\tilde{\mu}$) in decreasing order.

Definition 2.9. *Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \cdots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and*

$$\begin{cases} \text{(a)} & \tilde{\lambda} \neq \tilde{\mu} \ , \ \sum_{j=1}^r \tilde{\lambda}_j \geq \sum_{j=1}^r \tilde{\mu}_j \ \ (\textit{for all}\ \ r = 1,2,3,\cdots) \ , \textit{or} \\ \text{(b)} & \tilde{\lambda} = \tilde{\mu} \ , \ \sum_{j=1}^r k_j \geq \sum_{j=1}^r g_j \ \ (\textit{for all}\ \ r = 1,2,3,\cdots) \ . \end{cases} \tag{9}$$

Remark. The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

Example 2.10. *Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1,1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. In Uglov's order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\} = \{2, 2, -1\}$ and $\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\} = \{1, 1, 1\}$.*

We define a matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ by

$$\overline{|\lambda; s\rangle} = \sum_{\mu} a_{\lambda,\mu}(q)\,|\mu; s\rangle. \tag{10}$$

Then the matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ is unitriangular with respect to $\geq$, that is

$$\begin{cases} \text{(a)} \quad \text{if } a_{\lambda,\mu}(q) \neq 0 \text{ , then } |\lambda; s\rangle \geq |\mu; s\rangle, \\ \text{(b)} \quad a_{\lambda,\lambda}(q) = 1. \end{cases} \tag{11}$$

Thus, by the standard argument, the unitriangularity implies the following theorem.

Theorem 2.11. [Ugl00] *There exist unique bases* $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ *and* $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ *of* $F_q[s]$ *such that*

$$\begin{array}{llcl} \text{(i)} & \overline{G^+(\lambda; s)} = G^+(\lambda; s) & , & \overline{G^-(\lambda; s)} = G^-(\lambda; s) \\ \text{(ii)} & G^+(\lambda; s) \equiv |\,\lambda; s\rangle \mod q\,\mathcal{L}^+ & , & G^-(\lambda; s) \equiv |\,\lambda; s\rangle \mod q^{-1}\,\mathcal{L}^- \\ \textit{where} & \mathcal{L}^+ = \bigoplus_{\lambda\in\Pi^\ell} \mathbb{Q}[q]\,|\lambda; s\rangle & , & \mathcal{L}^- = \bigoplus_{\lambda\in\Pi^\ell} \mathbb{Q}[q^{-1}]\,|\lambda; s\rangle. \end{array}$$

Definition 2.12. *Define matrices* $\Delta^+(q) = (\Delta^+_{\lambda,\mu}(q))_{\lambda,\mu}$ *and* $\Delta^-(q) = (\Delta^-_{\lambda,\mu}(q))_{\lambda,\mu}$ *by*

$$G^+(\lambda; s) = \sum_{\mu} \Delta^+_{\lambda,\mu}(q)\,|\,\mu; s\rangle \quad , \quad G^-(\lambda; s) = \sum_{\mu} \Delta^-_{\lambda,\mu}(q)\,|\,\mu; s\rangle. \tag{12}$$

The entries $\Delta^\pm_{\lambda,\mu}(q)$ are called *q-decomposition numbers*. Note that q-decomposition numbers $\Delta^\pm(q)$ depend on n, ℓ and s. The matrices $\Delta^+(q)$ and $\Delta^-(q)$ are also unitriangular with respect to $\geq$.

It is known [Ugl00, Theorem 3.26] that the entries of $\Delta^-(q)$ are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A, and that they are polynomials in $p = -q$ with non-negative integer coefficients (see [KT02]).

3. A comparison of q-decomposition numbers

3.1. Sufficiently large and sufficiently small.

Definition 3.1. Let $s = (s_1, s_2, \cdots, s_\ell) \in \mathbb{Z}^\ell$ be a multi charge and $1 \leq j \leq \ell$.

(i). We say that the j-th component s_j of the multi charge s is *sufficiently large* for $|\lambda; s\rangle \in F_q[s]$ if

$$s_j - s_i \geq \lambda^{(i)}_1 \qquad \text{for all} \quad i = 1, 2, \cdots, \ell. \tag{13}$$

More generally, we say that s_j is sufficiently large for a q-wedge u_k if

$$s_j \geq c_r - nm_r \qquad \text{for all } r = 1, 2, \cdots, \tag{14}$$

where $k_r = c_r + n(d_r - 1) - n\ell m_r$, $(r = 1, 2, \cdots)$, $1 \leq c \leq n$ and $1 \leq d \leq \ell$ (see §2).

(ii). We say that s_j is *sufficiently small* for $|\lambda; s\rangle$ if

$$s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \qquad \text{for all} \quad i \neq j. \tag{15}$$

Note that the definition of sufficiently small depends only on the size of λ and the multi charge s. When we fix the multi charge s, we say that s_j is *sufficiently small* for N if

$$s_i - s_j \geq N \qquad \text{for all} \quad i \neq j. \tag{16}$$

Remark. If $|\lambda; s\rangle$ is 0-dominant in the sense of [Ugl00], that is

$$s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \qquad \text{for all} \quad i = 1, 2, \cdots, \ell - 1 \ ,$$

then s_1 is sufficiently large for $|\lambda; s\rangle$ and s_ℓ is sufficiently small for $|\lambda; s\rangle$.

Lemma 3.2. *If s_j is sufficiently large for $|\lambda; s\rangle$ and $|\lambda; s\rangle \geq |\mu; s\rangle$, then*
(i) $\lambda^{(j)} = \emptyset$,
(ii) *s_j is also sufficiently large for $|\mu; s\rangle$. In particular, $\mu^{(j)} = \emptyset$.*

Proof. It is clear that $\lambda^{(j)} = \emptyset$ by the definition.
Note that

$$\begin{aligned} s_j \text{ is sufficiently large for } |\lambda; s\rangle &\Leftrightarrow s_j - s_i \geq \lambda_1^{(i)} \qquad \text{for all } i = 1, 2, \cdots, \ell \\ &\Leftrightarrow s_j \geq \max\{\lambda_1^{(1)} + s_1, \cdots, \lambda_1^{(\ell)} + s_\ell\} = \tilde{\lambda}_1. \end{aligned}$$

If $|\lambda; s\rangle \geq |\mu; s\rangle$, then $\tilde{\lambda}_1 \geq \tilde{\mu}_1$ and so $s_j \geq \tilde{\mu}_1$. It means that s_j is sufficiently large for $|\mu; s\rangle$. □

Lemma 3.3. *Suppose that s_j is sufficiently small for $|\lambda; s\rangle$. If $|\lambda; s\rangle \geq |\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then $\lambda^{(j)} = \emptyset$.*

Proof. Suppose that $l(\lambda^{(j)}) \geq 1$. Then s_j is the minimal integer in the set $\{\mu_a^{(d)} + s_d \mid 1 \leq d \leq \ell,\ 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ because $\mu^{(j)} = \emptyset$ and s_j is the minimal integer in s. On the other hand, the minimal integer in the set $\{\lambda_a^{(d)} + s_d \mid 1 \leq d \leq \ell,\ 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ is greater than s_j because s_j is sufficiently small for $|\lambda; s\rangle$. Therefore $|\lambda; s\rangle \not\geq |\mu; s\rangle$. This is a contradiction. □

3.2. **Main results.** Now, we are ready to state our main theorems.

Theorem 3.4 ([Iij]). *Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently large for $|\lambda; s\rangle$, then*

$$\Delta^{\varepsilon}_{\lambda,\mu;s}(q) = \Delta^{\varepsilon}_{\check{\lambda},\check{\mu};\check{s}}(q), \tag{17}$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j-th component of λ (resp. μ, s).

Theorem 3.5 ([Iij]). *Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently small for $|\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then*

$$\Delta^{\varepsilon}_{\lambda,\mu;s}(q) = \Delta^{\varepsilon}_{\check{\lambda},\check{\mu};\check{s}}(q), \tag{18}$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j-th component of λ (resp. μ, s).

Example 3.6. (i) *If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = (\emptyset, (6))$, $\mu = (\emptyset, (5, 1))$, then s_1 is sufficiently large for $|\lambda; s\rangle$. Hence*

$$\Delta^{-}_{\lambda,\mu;s}(q) = \Delta^{-}_{\check{\lambda},\check{\mu};\check{s}}(q) = \Delta^{-}_{(6),(5,1);(-3)}(q) = -q^{-1}.$$

(ii) *If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = ((6), \emptyset)$, $\mu = ((5, 1), \emptyset)$, then s_2 is sufficiently small for $|\mu; s\rangle$. Hence*

$$\Delta^{-}_{\lambda,\mu;s}(q) = \Delta^{-}_{\check{\lambda},\check{\mu};\check{s}}(q) = \Delta^{-}_{(6),(5,1);(-3)}(q) = -q^{-1}.$$

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